A brief history of instability

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Outline of the talk

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2. An informal definition of instability
3. An example of an unstable system
4. Some conjectures by S.Smale and their counterexamples

Why do mathematicians study instability?

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When can an infinitesimally small experimental error induce an erroneous understanding of a system?

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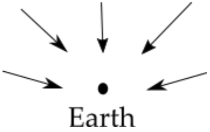
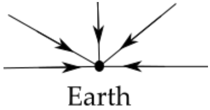
Answer (A. Andronov. and L. Pontrjagin, 1937):

- If a system is stable, we can understand it by doing experiments with very small errors
- We can never understand an unstable system by experiments, without a deeper mathematical study of the system

Gravitational field

The gravitational field is stable.





If the mass of the Earth changes by a small factor, objects are still going to be drawn to the Earth with a different acceleration.

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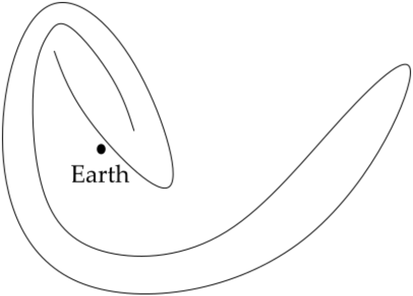

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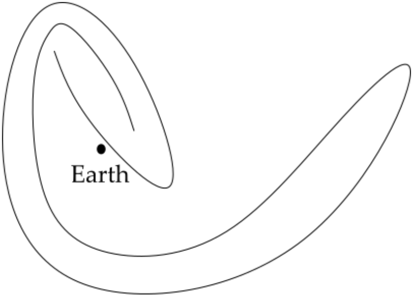

Question

How many systems in Nature and in Mathematics are unstable?

Continuous-time dynamical systems

Definition

Consider M a closed manifold of dimension n.

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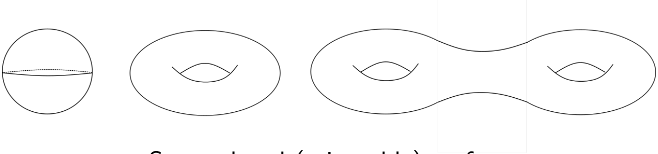
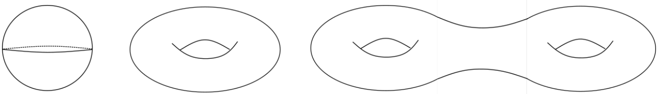


Some closed (orientable) surfaces

Continuous-time dynamical systems

Definition

Consider M a closed manifold of dimension n. A vector field is a smooth function from M to $\mathbb{R}^n$.



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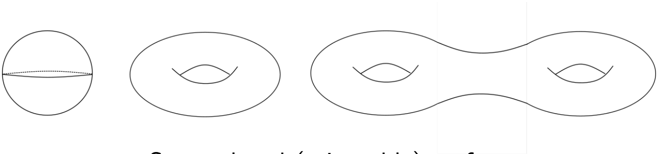
Consider M a closed manifold of dimension n. A vector field is a smooth function from M to $\mathbb{R}^n$.

A vector field corresponds to an ordinary differential equation of first order and vice versa



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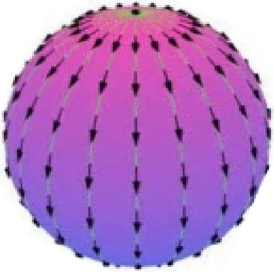

North-South vector field

Flows

Theorem (Cauchy-Lipschitz-Picard theorem)

To every smooth vector field X on a manifold M (therefore to every differential equation of order 1) we can associate a smooth flow, that is a function Φ_X that associates every couple $(x, t) \in M \times \mathbb{R}$ to a point $\Phi_X(x, t)$ in M.

Stability

Take a vector field X on a manifold M, its flow Φ_X and its associated differential equation

$$\dot{x} = F(x)$$

Denote by Y_ϕ the vector field associated to the differential equation

$$\dot{x} = F(x) + \phi(x)$$

where ϕ is an $\varepsilon > 0$ small smooth perturbation. Note Φ_{Y_ϕ} the flow associated to Y_ϕ.

Definition

The vector field X will be called stable if there exists $\varepsilon > 0$ such that for every ε-small perturbation ϕ the flow Φ_{Y_ϕ} is the same (up to a change of coordinates on M) as Φ_X.

An example of stable system

Theorem (Peixoto)

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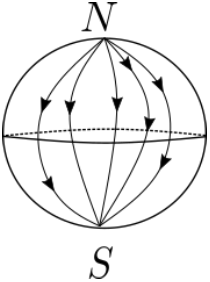

Before perturbation

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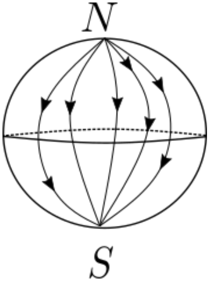

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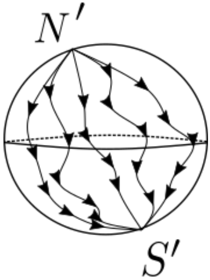
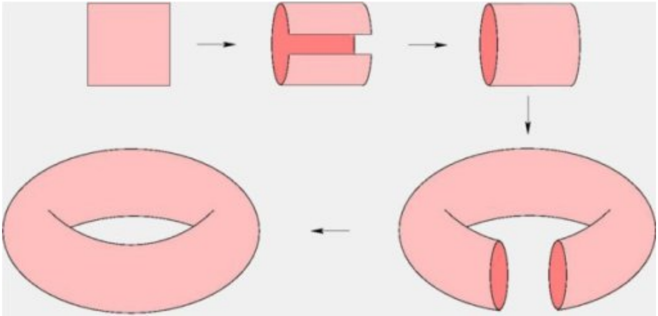


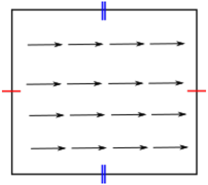
After perturbation

The torus as a square

A square with opposite sides identified is a torus!



An unstable system on the torus: the horizontal flow



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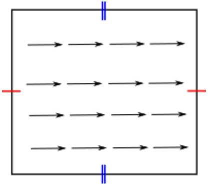
An unstable system on the torus: the horizontal flow

Every orbit of the horizontal flow is periodic.



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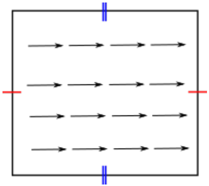

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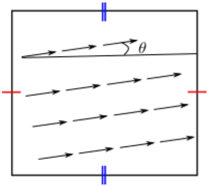

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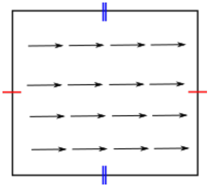

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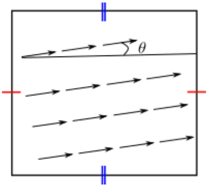

After perturbation

Proposition

If $\tan(\theta)$ is rational every orbit of the perturbed flow is periodic. If not, there is no periodic orbit!

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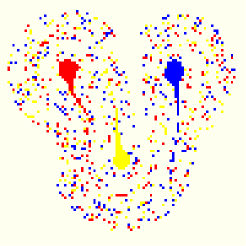
1. Every trajectory of a stable system is either at equilibrium or tries to get at an equilibrium state (i.e. become fixed or periodic)
2. Almost all systems in Mathematics and therefore in Nature are stable: we can change by a little bit the differential equation of an unstable system in order to turn it into a stable system

On the first conjecture of Smale

Theorem (Mañé, Robbin, Robbinson, Smale)

Every trajectory of a stable system is either at equilibrium or tries to get at an equilibrium state.
However, there exist equilibrium states (otherwise called non-trivial attractors) that are more complex and chaotic than a fixed or periodic orbit.



A realisation of the Plykin attractor by Yves Coudène

On the second conjecture of Smale

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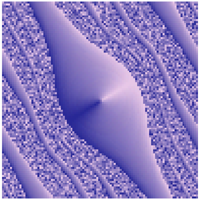
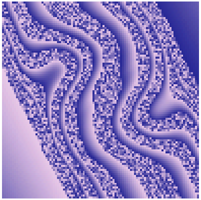
For manifolds of higher dimensions the second conjecture of Smale is wrong: there exist differential equations that despite small changes always describe unstable systems!

For each one of those systems, any attempt to understand them via experimentation would lead to erroneous results.

Thank you for your attention





Some more examples of attractors by Yves Coudène